Table 1 $H-J$ **critical points for various velocity profile families**

Profile family	Author	$H-J$ Critical point(s)
Falkner-Skan	Lees-Reeves[2]	$H = 0.2458, 0.2295$ 0.2237
Falkner-Skan	Lees-Reeves[2]	$H = 0.4570$
Turbulent similarity mean flow theory	Alber[7]	$H = 0.3175$
Turbulent similarity energy theory	Alber[7]	$H = 0.3250$

in assuming simultaneously an $H-J$ relation, a C_f model, and a C_D model, it is highly unlikely that both sides of Eq. (3) go to zero at the same time. Since $M_e(x)$ is presumed known, the $H-J$ relationship and/or the C_f model and/or the C_D model would have to be adjusted in an arbitrary manner in order to force both sides of Eq. (3) to zero at the same time. This is clearly distasteful. If $M_e(x)$ is the current distribution in an iteration scheme, the calculated $M_e(x)$ is presumably incorrect at intermediate steps and there would be no justification for adjusting the equations based on this wrong value. On the other hand, if $M_e(x)$ is known from measurements, it represents information of a higher degree of accuracy than is available for the $H-J$, C_f, and C_D models, which being only approximations, probably should not be expected to reproduce the exact flow in such a manner that both sides of Eq. (3) go to zero simultaneously.

The way out of this dilemma is to treat the problem as a strong interaction, with $M_e(x)$ unknown. The integrated continuity equation is introduced[1-4]

$$\left[H + \left(\frac{1+M_e}{m_e}\right)\right]\frac{d\delta^*}{dx} + \delta^*\frac{dH}{dx} + f\frac{\delta^*}{M_e}\frac{dM_e}{dx} = \frac{\tan\Theta}{m_e} \tag{4}$$

Solving Eqs. (1, 2, and 4) by Cramer's Rule

$$(\delta^*/M_e)\,dM_e/dx = N_1/D \tag{5}$$

$$\delta^*\,dH/dx = N_2/D \tag{6}$$

$$d\delta^*/dx = N_3/D \tag{7}$$

The forms of the N's and D are given elsewhere.[1-4] A singularity would occur in the preceding set of equations at $D = 0$. In this set, however, D is a function not only of the profile shape parameters H, J, and Z, but also of the external Mach number M_e. Apparently singularities in the preceding set only occur if M_e is greater than approximately 1.5. This is the well-known Crocco-Lees singularity and has actual physical significance.[1,8] Its appearance is not related to the $H-J$ critical point encountered previously.[6]

Since the set of Eqs. (5–7) does not contain the $H-J$ critical point, it is desirable to solve this set. This is done by prescribing Θ and solving the set. In an interaction calculation, the resulting solution is used in an inviscid calculation to obtain a better estimate for Θ. If the actual pressure (M_e) distribution is known, a trial and error solution is suggested. The Θ distribution is adjusted until the calculated $M_e(x)$ agrees with the measured value. It should be noted that this method is similar to that of Kuhn and Nielsen,[9] in which the skin friction is arbitrarily specified and $M_e(x)$ calculated from the boundary-layer equations. The main advantage of the approach outlined previously over that of Kuhn and Nielsen is that in a complete interaction calculation, an assumed Θ distribution can be continually updated from an inviscid analysis of the outer flow, while an assumed C_f distribution must be adjusted in some completely arbitrary manner.

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Hot-Wire Measurements in a Supersonic Jet at Low Reynolds Numbers

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Introduction

THERE is a growing community of researchers who believe that the turbulence in a jet has a degree of orderly structure whose characteristics are determined by the instability process. In the case of the supersonic jet, Tam[1] and Bishop et al.,[2] have theorized that the large-scale instability process is fundamental to the turbulent structure which contributes most substantially to the noise radiation from the jet. In fact, Tam[1] has developed an interesting theory which demonstrates how certain disturbance wavelengths are selectively amplified. Since in Tam's theory all nonlinear terms in the disturbance equations are neglected, it is strictly only applicable in laminar jets. The large velocity fluctuations found in the classical turbulent jet would normally preclude using linear stability theory in this flow situation, particularly if a high degree of accuracy is desirable. However, Landahl[3] and Crow and Champagne[4] have demonstrated that the modes of the linear stability problem are useful in describing the fluctuations in turbulent shear flows.

The theoretical analyses of Tam[1] and other supersonic jet noise researchers need experimental evidence against which to check their predictions. The particular need is information about the noise producing flow fluctuations within the jet. Some experimenters have used schlieren or holograph techniques to visualize these disturbances.[5,6] Our approach differs from theirs not only in the measurement technique but also in the fact that ours focuses on the classical laminar instabilities. By exhausting the jet into a vacuum chamber, low enough Reynolds numbers are obtained so that there are a few diameters of laminar flow in the jet.

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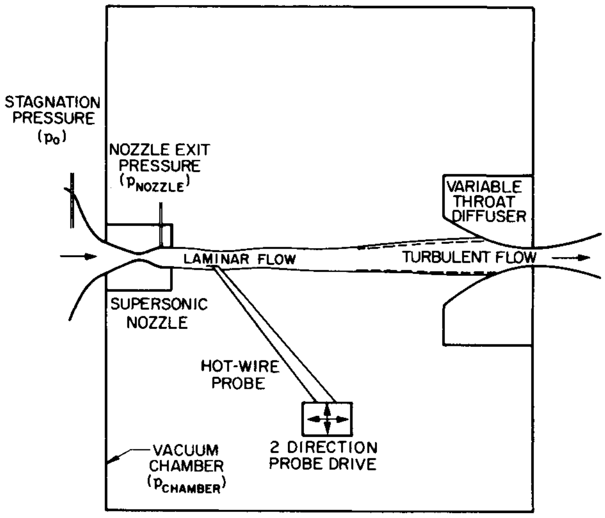

Fig. 1 Schematic of supersonic jet test section.

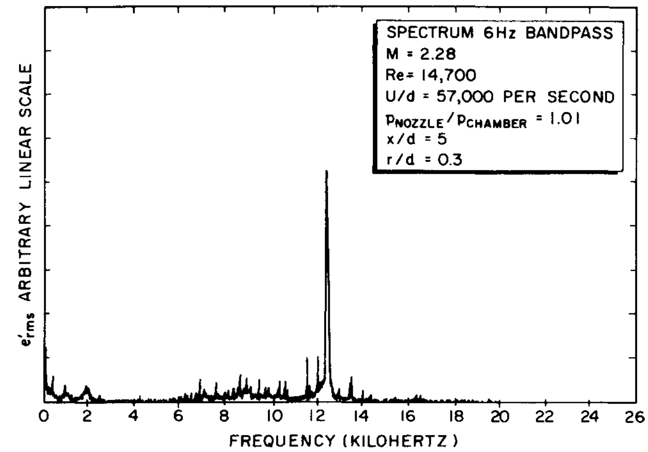

Fig. 3 Spectrum of hot-wire voltage fluctuation at $Re = 14{,}700$.

Experimental Method

The basic facility is an axisymmetric freejet of exit diameter 9.52 mm (see Fig. 1). The throat diameter of 5.46 mm gives a nozzle area ratio of 3.036 which yields a Mach number of 2.65 in inviscid flow. The nozzle has a simple conical contour with a 2.28° half angle. The Reynolds number range of the present experiments is $Re = 14,700$–$44,100$. This contrasts with the Reynolds number $Re = 1.24 \times 10^6$ one obtains with a similar Mach number jet exhausting to atmosphere. The stagnation temperature for all measurements is atmospheric temperature.

The basic sensing elements of the hot-wire probes are Disa Model 55A53 subminiature probes epoxied to the upper edge of slender brass wedges. The probes are operated in the constant temperature mode using Disa Model 55A01 electronics. Pitot pressure and static pressure probes are used to establish the mean flow quantities in the jet. At the low Reynolds numbers used in the present experiments, measured Mach numbers range from 2.20 to 2.40 on the jet center line near the exit plane (average $M = 2.3$). More details on the experimental method can be found in McLaughlin and McColgan.[7]

Experimental Results and Discussion

Figure 2 presents spatial distributions of hot-wire rms voltage fluctuation amplitudes at a Reynolds number of 14,700. We know of no other hot-wire measurements in a jet which is laminar for a number of diameters: however, there is a good deal of experimental evidence on the transition of supersonic wakes[8,9] which have similar flowfields to jets (making the appropriate coordinate transformation). A comparison of the results of the hot-wire measurements at $Re = 14{,}700$ (Fig. 2) with measurements of supersonic wakes clearly indicates transition from laminar to turbulent flow in the jet.

Using the procedure similar to Behrens and Ko[9] of interpreting hot-wire voltage fluctuations in terms of mass flux fluctuations, we are able to determine that the peak mass flux fluctuations at one diameter downstream of the jet exit are less than 3% of the local mean mass flux. This would indicate initially laminar flow. The fluctuations grow and spread through the jet so that at 9 diam downstream, the jet appears fully turbulent.

The spectra of the hot-wire fluctuations provide complementary and additional information about the transition process from laminar to turbulent flow. Figure 3 is a spectrum of the hot-wire fluctuations in the maximum fluctuation region of the shear annulus 5 diam downstream of the jet exit. There is a very clear pure tone in this spectrum of approximately 12,000 Hz (Strouhal Number of $St \approx 0.21$ using the centerline exit velocity and exit diameter to form the characteristic frequency). The single frequency nature of the fluctuations is similar to that found in the instability region of the axisymmetric supersonic wake[8] lending more support to our interpretation that the jet is initially laminar. In comparing with the theory of Tam,[1] it is encouraging that for a Mach number 2.3 jet, Tam's theory predicts a dominant oscillation of $St \approx 0.20$.

Over the Reynolds number range of this study, there was a gradual tendency for the frequency spectra of the hot-wire signal

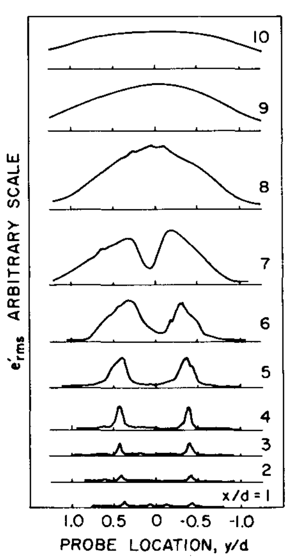

Fig. 2 Profiles of rms hot-wire voltage fluctuations at various downstream locations in the perfectly expanded jet: ($Re = 14{,}700$, $p_N/p_{CH} = 1.01$).

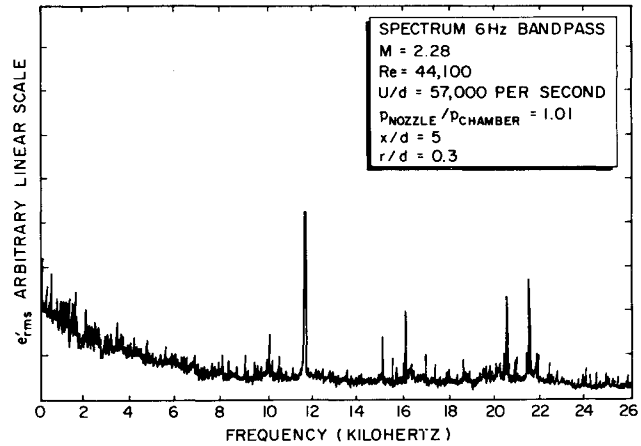

Fig. 4 Spectrum of hot-wire voltage fluctuation at $Re = 44{,}100$.

to include a broader range of frequencies as the Reynolds number of the jet was increased. In fact, at the highest Reynolds number, spectra were obtained such as shown in Fig. 4 with a broad frequency content including numerous discrete frequency peaks. It appears that over the Reynolds number range of these experiments there remained a good deal of orderly structure in the fluctuations even where simple amplitude measurements would indicate a turbulent flow situation. In view of these results, and those of Crow and Champagne,[4] it is reasonable to expect some orderly structure in the oscillations of supersonic jets of much higher Reynolds numbers.

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Series Expansion of the Eccentricity for Near Parabolic Orbits

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Introduction

THE determination of the elements of a near parabolic orbit is difficult because of the slow convergence of the series involved. This Note provides a convergent series to compute the eccentricity of a near parabolic orbit in terms of the pericenter distance q and the time of flight t through an angular distance of $\pi/2$ rad from the pericenter (Fig. 1). For a planet flyby, these quantities can be accurately determined by onboard radar measurements and observations. In the figure, the elliptic, parabolic, and hyperbolic orbit, all having the same pericenter, are denoted by E, P, and H, respectively. The elements of the elliptic, or hyperbolic, orbit are to be computed, while the parabolic orbit is introduced as a reference orbit. The key equation for the expansions is a hypergeometric equation.

The time of flight from the pericenter, along an elliptic orbit, is given by

$$(\mu/a^3)^{1/2}\, t = E - e\sin E \tag{1}$$

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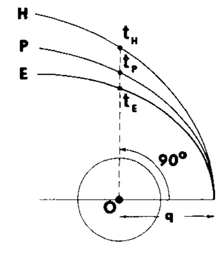

Fig. 1 Nomenclature.

and for a hyperbolic orbit

$$(\mu/a^3)^{1/2}\, t = e\sinh H - H \tag{2}$$

while for a parabolic orbit, we have

$$(\mu/2q^3)^{1/2}\, t = \tan(v/2) + \tfrac{1}{3}\tan^3(v/2) \tag{3}$$

where μ is the gravitational constant, a is the semi-major axis, v is the true anomaly, E is the eccentric anomaly, and H the hyperbolic anomaly. Since

$$\cos v = (\cos E - e)/(1 - e\cos E),$$

$$\cos v = (e - \cosh H)/(e\cosh H - 1) \tag{4}$$

when $v = \pi/2$, we have

$$\begin{aligned} e &= \cos E, \quad \sin^2(E/2) = (1-e)/2 \\ e &= \cosh H, \quad \sinh^2(H/2) = (e-1)/2 \end{aligned} \tag{5}$$

Then, the time of flight from the pericenter to one end of the semi-latus rectum, along an elliptic orbit, is

$$t_E = (2q^3/\mu)^{1/2}(E - \sin E\cos E)/4\sin^3(E/2) \tag{6}$$

and for a hyperbolic orbit

$$t_H = (2q^3/\mu)^{1/2}(\sinh H\cosh H - H)/4\sinh^3(H/2) \tag{7}$$

On the other hand, if the flight is along the parabolic orbit, we have

$$t_P = \tfrac{4}{3}(2q^3/\mu)^{1/2} \tag{8}$$

Assume that the pericenter distance q, and the time of flight t to one end of the semi-latus rectum along either an elliptic or hyperbolic orbit are known. Hence t_P can also be computed from Eq. (8).

Let

$$\phi = \tfrac{5}{2}(\mu/2q^3)^{1/2}(t_P - t) \tag{9}$$

Then, from Eqs. (6–8)

$$\phi = \tfrac{10}{3} - 5(E - \sin E\cos E)/8\sin^3(E/2) \tag{10}$$

for the elliptic case, or

$$\phi = \tfrac{10}{3} - 5(\sinh H\cosh H - H)/8\sinh^3(H/2) \tag{11}$$

for the hyperbolic case.

Since ϕ is known, the transcendental equation [(10) or (11)] could be solved for E or H and subsequently the eccentricity of the orbit could be obtained from Eq. (5). For near parabolic orbit, E and H are small and a direct computation is sensitive to error. We propose to find a convergent series for the computation of E and H and subsequently, of the eccentricity e, in terms of ϕ. For this purpose, let

$$\begin{aligned} \theta &= \sin^2(E/2), \quad \text{elliptic case} \\ \theta &= -\sinh^2(H/2), \quad \text{hyperbolic case} \end{aligned} \tag{12}$$

By taking the derivative of ϕ with respect to E, we have

$$\tan(E/2)\, d\phi/dE + \tfrac{3}{2}\phi = 5 - 5\cos(E/2)$$

Using Eq. (12) to change the independent variable from E to θ

$$\theta(d\phi/d\theta) + \tfrac{3}{2}\phi = 5 - 5\cos(E/2) \tag{13}$$

By taking the derivative of this equation with respect to θ, we have

$$\theta(1-\theta)\, d^2\phi/d\theta^2 + \tfrac{5}{2}(1-\theta)\, d\phi/d\theta = \tfrac{5}{2}\cos(E/2) \tag{14}$$

By eliminating $\cos(E/2)$ between the two equations, we have the hypergeometric equation[1]

$$\theta(1-\theta)\, d^2\phi/d\theta^2 + (\tfrac{5}{2} - 2\theta)\, d\phi/d\theta + \tfrac{3}{4}\phi = \tfrac{5}{2} \tag{15}$$

A similar process in the hyperbolic case results in the identical equation.